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Permissible growth rates for Birkhoff type universal harmonic functions

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Abstract

A harmonic function H on $\mathbb{R}^n\,(n \geqslant 2)$ is said to be universal (in the sense of Birkhoff) if its set of translates $\{x \mapsto H(a+x) : a \in \mathbb{R}^n\}$ is dense in the space of all harmonic functions on $\mathbb{R}^n$ with the topology of local uniform convergence. The main theorem includes the result that such functions, H, can have any prescribed order and type. The growth result is compared with a similar known theorem for G.D. Birkhoff's universal holomorphic functions and contrasted with known growth theorems for MacLane-type universal harmonic and holomorphic functions.

Keywords: Harmonic; Universal; Growth; Translation; Density

1. Introduction

Let $\mathcal{E}$ denote the space of all entire (holomorphic) functions on $\mathbb{C}$, the complex plane, and $\mathcal{H}_n$ the space of all harmonic functions on $\mathbb{R}^n$, where $n \geqslant 2$. These spaces are endowed with the topology of local uniform convergence (the compact-open topology). In 1929 G. Birkhoff [11] showed that there exist elements of $\mathcal{E}$ whose translates are dense in $\mathcal{E}$; we call such elements *universal holomorphic* functions. Thus F is a universal holomorphic function if $F \in \mathcal{E}$ and for every compact subset K of $\mathbb{C}$, every $f \in \mathcal{E}$ and every $\varepsilon > 0$ there exists $a \in \mathbb{C}$ such that $|F(z+a) - f(z)| < \varepsilon$ for all z in K. Similarly, Dzagnidze [17] showed

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that there exist elements of $\mathcal{H}_n$ whose translates are dense in $\mathcal{H}_n$; we call these *universal harmonic* functions. Universal functions can easily be shown to exist by a recursive construction using classical approximation theorems, namely Runge's theorem in the holomorphic case and Walsh's theorem in the harmonic case (for which see e.g. [18, Theorem 8.4]). Alternatively, modern theorems about tangential approximation on unbounded sets can be used to give quite short existence proofs (see e.g. [5, Theorem 11.1] and [7] for the harmonic case). It is obvious from the definitions that universal functions form dense subsets of the spaces $\mathcal{E}$ and $\mathcal{H}_n$. Much more is true: they form residual subsets; that is to say, the non-universal functions form first category subsets of the Baire spaces $\mathcal{E}$ and $\mathcal{H}_n$. This was proved by Duios Ruis [15] in the holomorphic case, and his proof can be mimicked in the harmonic case (see e.g. [5, Theorem 11.2]). Also, each of the spaces $\mathcal{E}$ and $\mathcal{H}_n$ contains an infinite-dimensional closed vector subspace whose elements, except for 0, are universal; this result is due to Bernal and Montes [10] in the holomorphic case and Bonilla [12] in the harmonic case. Duios Ruis [16] has sketched a proof of the existence of universal holomorphic functions of arbitrarily slow transcendental growth; a precise statement is given in Section 4 below. More general and more easily accessible results are given by Chan and Shapiro [14]. In the harmonic case it is easy to modify standard existence proofs to produce universal functions of arbitrarily rapid growth. In this note we show that universal harmonic functions can also have slow growth.

Theorem. *Let $\phi\colon [0, +\infty) \to (0, +\infty)$ be a continuous increasing function such that*

$$\frac{\log \phi(t)}{(\log t)^2} \to +\infty \quad \text{as } t \to +\infty. \tag{1}$$

There exists a universal harmonic function H in $\mathcal{H}_n$, where $n \geqslant 2$, such that $|H(x)| \leqslant \phi(\|x\|)$ for all x in $\mathbb{R}^n$ and

$$\limsup_{x\to\infty} H(x)/\phi(\|x\|) = 1. \tag{2}$$

Thus there are universal harmonic functions of all orders and types, including order 0. Hitherto it seems to have been uncertain whether such functions could even be of finite order. It remains an open question whether (1) can be relaxed: can the exponent 2 of $\log t$ be reduced, perhaps to 1?

Another type of universality was introduced by G.R. MacLane [25], who showed that there exist functions F in $\mathcal{E}$ for which the sequence $(F^{(j)})$ of derivatives is dense in $\mathcal{E}$. Similarly there are harmonic functions H whose partial derivatives form a dense subset of $\mathcal{H}_n$. In contrast with the Birkhoff type universal functions discussed in this note, MacLane's universal functions and their harmonic analogues cannot have very slow growth: they can be of exponential type 1 but not of exponential type less than 1. A precise description of the permissible growth rates of MacLane's functions is given by Grosse-Erdmann [19] (see also [21] and [9]), and corresponding results for their harmonic analogues are given by Aldred and Armitage [2]; see also [3].

Section 2 below contains a sequence of lemmas leading up to the proof of the Theorem, which is given, for the case $n \geqslant 3$, in Section 3. The case $n = 2$ is treated separately in Section 4.

The reader is referred to [20] and [22] for updated surveys about many kinds of universality and general properties. For discussions of growth rates in relation to other classes of universal functions, see [4], [23] and [24].

2. Auxiliary results

2.1.

For a subset E of $\mathbb{R}^n$ and a positive number r, we define $rE = \{rx : x \in E\}$. As usual E^0, $\overline{E}$ and ∂E denote the interior, closure and boundary of E, respectively. Our first lemma is a minor extension of a Schwarz lemma due to Bagby and Levenberg [8].

Lemma 1. *Let Ω be a domain in $\mathbb{R}^n$, where $n \geqslant 2$, such that $\mathbb{R}^n \backslash \Omega$ is compact. Let L be a closed subset of Ω, and let r be a positive number. There exist positive constants C_1 and p, depending only on Ω and L, with the following property: if u is a function continuous on $r\overline{\Omega}$ and harmonic on $r\Omega$, and there exists an integer $j > n-2$ such that $u(x) = O(\|x\|^{-j})$ as $x \to \infty$, then*

$$\max_{rL} |u| \leqslant C_1 e^{-pj} \max_{r\partial\Omega} |u|.$$

With $r = 1$, Lemma 1 is a paraphrase of [8, Corollary 2.5], and the general result follows by a simple dilation argument.

2.2.

The open ball of centre x and radius r in $\mathbb{R}^n$ is denoted by $B(x, r)$. We denote by $y(t)$ the point of $\mathbb{R}^n$ with coordinates $(10t, 0, \ldots, 0)$ and define

$$T(t, r) = \overline{B(0, r)} \cup \overline{B(y(t), r)},$$

where $0 < r \leqslant 5t$. We write $\mathcal{H}_{j,n}$ for the space of all harmonic polynomials of degree at most j on $\mathbb{R}^n$. For a compact subset K of $\mathbb{R}^n$ and a bounded function u on K, we define

$$d_j(u, K) = \inf \left\{ \sup_K |u - P| : P \in \mathcal{H}_{j,n} \right\}.$$

Lemma 2. *There exist positive numbers C_2 and p, depending only on n, with the following property. If $r > 0$ and j, k are positive integers, and if u is a harmonic function on $\mathbb{R}^n$, where $n \geqslant 3$, satisfying*

$$|u| < r^{k+1} \quad \text{on } B(y(r), 5r), \tag{3}$$

then

$$d_j(\tilde{u}, T(r, r)) < C_2 r^{k+1} e^{-pj}, \tag{4}$$

where $\tilde{u} = 0$ on $\overline{B(0, r)}$ and $\tilde{u} = u$ on $\overline{B(y(r), r)}$.

A small modification to Lemma 2 is required in the case $n = 2$; this is discussed in Section 4 below. We start the proof of Lemma 2 with the observation that if $u \in \mathcal{H}_n$ and satisfies (3), then

$$|\partial u/\partial x_m| \leqslant 2nr^k \quad \text{on } B(y(r), 4r) \ \ (m = 1, \ldots, n); \tag{5}$$

this follows easily from Harnack's inequalities (see e.g. [6, p. 14]). Let $\psi_o\colon \mathbb{R}^n \to [0, 1]$ be an infinitely differentiable function such that $\psi_o = 1$ on $B(0, 3)$ and $\operatorname{supp} \psi_o \subset B(0, 4)$. Then there exists a positive constant C, depending only on ψ_o and hence only on n, such that

$$\left|\frac{\partial \psi_o}{\partial x_m}\right| + \left|\frac{\partial^2 \psi_o}{\partial x_m^2}\right| \leqslant C \quad \text{on } \mathbb{R}^n \ \ (m = 1, \ldots, n).$$

We define $\psi(x) = \psi_o((x - y(r))/r)$. Then $\psi = 1$ on $B(y(r), 3r)$ and $\operatorname{supp} \psi \subset B(y(r), 4r)$. Also

$$\left|\frac{\partial \psi}{\partial x_m}\right| \leqslant Cr^{-1}, \quad \left|\frac{\partial^2 \psi}{\partial x_m^2}\right| \leqslant Cr^{-2} \quad \text{on } \mathbb{R}^n \ \ (m = 1, \ldots, n). \tag{6}$$

We define $V = \psi u$. From now on our proof is closely modelled on the proof of [8, Theorem 3.1]. Let j be a fixed positive integer. As is remarked in [8], the Hahn-Banach theorem and the Riesz representation theorem imply the existence of a signed measure μ of total variation 1 such that $\operatorname{supp} \mu \subseteq T(r, r)$,

$$\int P \, d\mu = 0 \quad \text{for all } P \in \mathcal{H}_{j,n} \tag{7}$$

and

$$d_j(\tilde{u}, T(r, r)) = \int V \, d\mu.$$

Since V is infinitely differentiable on $\mathbb{R}^n$ and has compact support,

$$V(x) = c_n \int \|x - y\|^{2-n} \Delta V(y) \, d\lambda(y) \quad (x \in \mathbb{R}^n),$$

where λ denotes n-dimensional Lebesgue measure, Δ is the Laplacian operator on $\mathbb{R}^n$, and the constant c_n is given by $c_n = ((2 - n)\sigma_n)^{-1}$, where σ_n denotes the $(n - 1)$-dimensional surface area of $\partial B(0, 1)$; see e.g. [6, Lemma 4.3.6]. Integrating with respect to μ and using Fubini's theorem to justify a change of order of integration, we find that

$$d_j(\tilde{u}, T(r, r)) = \int V \, d\mu = \int v \Delta V \, d\lambda, \tag{8}$$

where

$$v(x) = c_n \int \|x - y\|^{2-n} \, d\mu(y). \tag{9}$$

(Our proof of (8) is essentially the same as the more succinct argument given in [8, pp. 12–13].)

Our first step in estimating the integral in (8) is to estimate ΔV. We have

$$|\Delta V| \leqslant |u\Delta\psi| + 2\sum_{m=1}^{n}\left|\frac{\partial\psi}{\partial x_m}\frac{\partial u}{\partial x_m}\right|.$$

Since supp $V \subset B(y(r), 4r)$ it follows from (3), (5) and (6) that

$$|\Delta V| \leqslant Cnr^{k-1} + 4Cn^2r^{k-1} = C'r^{k-1}, \tag{10}$$

where $C' = Cn(4n+1)$. Next we estimate v. Since supp $\mu \subset T(r,r)$ and μ has total variation 1, we see that

$$|v| \leqslant c_n r^{2-n} \quad \text{on } \partial T(r, 2r). \tag{11}$$

Now, with a view to applying Lemma 1, we consider the behaviour of $v(x)$ as $x \to \infty$. It is well known (see e.g. [5, p. 49] or [18, p. 75]) that

$$\|x-y\|^{2-n} = \sum_{m=0}^{\infty} b_m P_m(x,y) \quad (\|x\| > \|y\|), \tag{12}$$

where

$$b_m = \binom{m+n-3}{m} = O(m^{n-3}) \quad \text{as } m \to \infty$$

and the functions P_m have the following properties: for each x in $\mathbb{R}^n\backslash\{0\}$ the function $P_m(x,\cdot)$ is a homogeneous harmonic polynomial of degree m; for each y in $\mathbb{R}^n$ the function $P_m(\cdot, y)$ is harmonic on $\mathbb{R}^n\backslash\{0\}$, and

$$|P_m(x,y)| \leqslant \|x\|^{2-n-m}\|y\|^m \quad (x, y \in \mathbb{R}^n, x \neq 0).$$

Since supp $\mu \subseteq T(r,r) \subset B(0, 11r)$, we find, using (7) and the properties of P_m, that when $\|x\| > 11r$

$$\begin{aligned}
|v(x)| &\leqslant c_n \sum_{m=j+1}^{\infty}\left|b_m \int P_m(x,y)\,d\mu(y)\right| \\
&\leqslant c_n \sum_{m=j+1}^{\infty} b_m \sup\{|P_m(x,y)|\colon y \in T(r,r)\} \\
&= O\left(\|x\|^{2-n}\sum_{m=j+1}^{\infty} m^{n-3}(11r/\|x\|)^m\right) \\
&= O(\|x\|^{1-n-j}) \quad \text{as } x \to \infty.
\end{aligned} \tag{13}$$

We take $\Omega = \mathbb{R}^n\backslash T(1,2)$ and $L = \mathbb{R}^n\backslash T(1,3)^\circ$ in Lemma 1 and note that $r\Omega = \mathbb{R}^n\backslash T(r,2r)$ and $rL = \mathbb{R}^n\backslash T(r,3r)^\circ$. Let C_1 and p be the constants in Lemma 1 corresponding to this choice of Ω and L. Thus C_1 and p now depend only on n. Since v is harmonic on $\mathbb{R}^n\backslash T(r,r)$ and satisfies (11) and (13), it follows from Lemma 1 that

$$|v(x)| \leqslant C_1 c_n r^{2-n} e^{-p(n+j-1)} \quad (x \in \mathbb{R}^n\backslash T(r,3r)^\circ). \tag{14}$$

Since

$$\operatorname{supp} \Delta V \subset B(y(r), 4r) \backslash B(y(r), 3r) \subset \mathbb{R}^n \backslash T(r, 3r)^{\circ},$$

we find, using (8), (10) and (14), that

$$\begin{aligned} d_j(\tilde{u}, T(r, r)) &\leqslant C' r^{k-1} C_1 c_n r^{2-n} e^{-p(n+j-1)} \lambda(B(0, 4r) \backslash B(0, 3r)) \\ &< C_2 r^{k+1} e^{-pj}, \end{aligned}$$

where C_2 depends only on n.

2.3.

Lemma 3. *If $r > 0$ and $P \in \mathcal{H}_{k,n}$ for some positive integer k, then*

$$|P(x)| < C_3 (k+1)^{n/2} (\|x\|/r)^k \sup_{\partial B(0,r)} |P| \quad (\|x\| > r),$$

where C_3 is a constant depending only on n.

To prove this, let $P = \sum_{j=0}^{k} P_j$, where P_j is a homogeneous harmonic polynomial of degree j. Then it is known that

$$\sup_{\partial B(0,r)} |P_j| \leqslant C_3 (j+1)^{(n-2)/2} \sup_{\partial B(0,r)} |P| \quad (j = 0, 1, \ldots, k), \tag{15}$$

where C_3 depends only on n. In the case where $r = 1$, the inequality (15) follows from an inequality of Brelot and Choquet [13, Proposition 4] (or see e.g. [6, Corollary 2.3.8]), which shows that $\sup_{\partial B(0,1)} |P_j| \leqslant (\delta_j M(P_j))^{1/2}$, where $M(P_j)$ denotes the mean value of P_j^2 on $\partial B(0, 1)$ and δ_j is the dimension of the vector space of all homogeneous harmonic polynomials of degree j on $\mathbb{R}^n$. We note that $\delta_j = O(j^{n-2})$ as $j \to \infty$; this follows easily from an explicit formula for δ_j (for which, see e.g. [6, Corollary 2.1.4]). Details of the proof of (15) with $r = 1$ are given in [1]. The general case is obtained by a dilation argument. Using (15) and the homogeneity of P_j, we find that if $\|x\| > r$, then

$$\begin{aligned} |P(x)| \leqslant \sum_{j=0}^{k} |P_j(x)| &= \sum_{j=0}^{k} (\|x\|/r)^j |P_j(rx/\|x\|)| \\ &\leqslant \sum_{j=0}^{k} (\|x\|/r)^j C_3 (j+1)^{(n-2)/2} \sup_{\partial B(0,r)} |P| \\ &< C_3 (k+1)^{n/2} (\|x\|/r)^k \sup_{\partial B(0,r)} |P|. \end{aligned}$$

2.4.

Lemma 4. *Let ϕ be as in the Theorem, and let P, Q be harmonic polynomials on $\mathbb{R}^n$, where $n \geqslant 3$. If $\varepsilon > 0$ and r is sufficiently large, then there exists a harmonic polynomial F such that*

$$|F| < \varepsilon \quad \text{on} \quad B(0,r),$$

$$|F(x) - (Q(x) + P(x - y(r)))| < \varepsilon \quad (x \in B(y(r), r))$$

and

$$|F(x)| < \varepsilon\phi(\|x\|) \quad (x \in \mathbb{R}^n). \tag{16}$$

To prove this, we note first there exist a positive number A and an integer $k \geqslant 2$ such that

$$|P(x)| + |Q(x)| < A(1 + \|x\|)^k \quad (x \in \mathbb{R}^n).$$

Then, if r is sufficiently large,

$$|P(x - y(r))| + |Q(x)| < A(1 + 5r)^k + A(1 + 15r)^k < r^{k+1} \quad (x \in B(y(r), 5r)). \tag{17}$$

We suppose without loss of generality that $\phi(0) < 1$, $\varepsilon < 1$ and $C_3 > 1$. We now fix a number $r > e$ so large that (17) holds together with the following inequalities:

$$\phi(t) > t^{(2k \log t)/p} \quad (t \geqslant r), \tag{18}$$

$$C_2 e r^{1-k} < \varepsilon\phi(0)/C_3, \tag{19}$$

$$(1 + (2k/p) \log r)^{n/2} r^{1-(2k \log r)/p} < 1. \tag{20}$$

Let m be the greatest integer not exceeding $(2k \log r)/p$. By Lemma 2 there is an element F of $\mathcal{H}_{m,n}$ such that

$$|F| < C_2 r^{k+1} e^{-pm} \quad \text{on } B(0,r) \tag{21}$$

and

$$|F(x) - (Q(x) + P(x - y(r)))| < C_2 r^{k+1} e^{-pm} \quad (x \in B(y(r), r)). \tag{22}$$

Now $e^{-pm} < er^{-2k}$ by our choice of m. Hence by (19),

$$|F| < \varepsilon\phi(0)/C_3 \quad \text{on } B(0,r) \tag{23}$$

and

$$|F(x) - (Q(x) + P(x - y(r)))| < \varepsilon\phi(0)/C_3 < \varepsilon \quad (x \in B(y(r), r)).$$

By (23) and Lemma 3, and then by definition of m, (20) and (18),

$$\begin{aligned} |F(x)| &< \varepsilon(m+1)^{n/2}(\|x\|/r)^m \qquad (\|x\| > r) \\ &< \frac{\varepsilon(1+(2k\log r)/p)^{n/2}\|x\|^{(2k\log r)/p}}{r^{-1+(2k\log r)/p}} \\ &< \varepsilon\|x\|^{(2k\log\|x\|)/p} < \varepsilon\phi(\|x\|), \end{aligned}$$

which, together with (23), shows that (16) holds.

2.5.

Lemma 5. *Let $\phi\colon [0,+\infty) \to (0,+\infty)$ be a continuous increasing function such that $\log\phi(t)/\log t \to +\infty$ as $t \to \infty$. Let ε, η, r be positive numbers with $\eta < 1$. Then there exists a harmonic polynomial G on $\mathbb{R}^n$ such that $|G| < \varepsilon$ on $B(0,r)$ and*

$$|G(x)| \leqslant (1-\eta)\phi(\|x\|)$$

for all x in $\mathbb{R}^n$ with equality for some x.

By Walsh's theorem on harmonic approximation (see e.g. [18, p. 8]) there exists a harmonic polynomial G_o such that $|G_o| < \varepsilon$ on $B(0,r)$ and $G_o(x_o) > \phi(\|x_o\|)$ for some x_o in $\mathbb{R}^n$. Now the function $G_o(x)/\phi(\|x\|)$ tends to 0 as $x \to \infty$ and therefore attains a maximum value c, say, at some point y. We define $G = (1-\eta)G_o/c$. Since $c > 1$, we see that $|G| \leqslant |G_o| < \varepsilon$ on $B(0,r)$. Also, for each x in $\mathbb{R}^n$

$$|G(x)| = (1-\eta)c^{-1}|G_o(x)| \leqslant (1-\eta)\phi(\|x\|)$$

with equality when $x = y$.

3. Proof of the Theorem for $n \geqslant 3$

3.1.

Throughout this section we suppose that $n \geqslant 3$. The Theorem will follow without difficulty from the following result, which will be proved by using Lemmas 4 and 5.

Proposition. *Let ϕ be as in the Theorem, and let (P_m) be a sequence of harmonic polynomials. There exist sequences (ξ_m), (η_m) of points on the positive x_1-axis of $\mathbb{R}^n$ such that*

$$0 < 2\|\xi_m\| < \|\eta_m\| < 10^{-1}\|\xi_{m+1}\|$$

and harmonic polynomials H_m such that

$$\begin{aligned} &|H_m| < 2^{-m}\phi(0) \quad \textit{on } B(0, 10^{-1}\|\xi_m\|), \\ &|(H_1 + \cdots + H_m)(x) - P_m(x-\xi_m)| < 2^{-m} \quad (x \in B(\xi_m, 10^{-1}\|\xi_m\|)), \\ &|(H_1 + \cdots + H_m)(x)| < (1-2^{-m-2})\phi(\|x\|) \quad (x \in \mathbb{R}^n), \end{aligned} \tag{24}$$

and

$$|(H_1 + \cdots + H_m)(\eta_m)| > (1 - 2^{-m}\phi(\|\eta_m\|)).$$

In proving the Proposition, we suppose without loss of generality that $\phi(0)<1$. By Lemma 4 there exist a harmonic polynomial F_1 and a positive number r such that

$$|F_1| < 2^{-2}\phi(0) \quad \text{on } B(0,r), \tag{25}$$

$$|F_1(x) - P_1(x - y(r))| < 2^{-3}\phi(0) \quad (x \in B(y(r), r)), \tag{26}$$

and

$$|F_1(x)| < 2^{-3}\phi(\|x\|) \quad (x \in \mathbb{R}^n). \tag{27}$$

We choose $\xi_1 = y(r)$. By Lemma 5 there exists a harmonic polynomial G_1 such that

$$|G_1| < 2^{-2}\phi(0) \quad \text{on } B(0, 2\|\xi_1\|) \tag{28}$$

and

$$|G_1(x)| \leqslant (1 - 2^{-2})\phi(\|x\|) \quad (x \in \mathbb{R}^n) \tag{29}$$

with equality at some point η_1. Clearly $\|\eta_1\| > 2\|\xi_1\|$ and we may suppose (by composing G_1 with a rotation of $\mathbb{R}^n$) that η_1 lies on the positive x_1-axis. Let $H_1 = F_1 + G_1$. Then, by (25) and (28),

$$|H_1| \leqslant |F_1| + |G_1| < 2^{-1}\phi(0) \quad \text{on } B(0, 10^{-1}\|\xi_1\|),$$

and

$$\begin{aligned} |H_1(x) - P_1(x - \xi_1)| &\leqslant |F_1(x) - P_1(x - \xi_1)| + |G_1(x)| \\ &\leqslant 2^{-3}\phi(0) + 2^{-2}\phi(0) < 2^{-1}\phi(0) \quad (x \in B(\xi_1, 10^{-1}\|\xi_1\|)) \end{aligned}$$

by (26) and (28). Also, for all x in $\mathbb{R}^n$,

$$|H_1(x)| \leqslant |F_1(x)| + |G_1(x)| < 2^{-3}\phi(\|x\|) + (1 - 2^{-2})\phi(\|x\|) = (1 - 2^{-3})\phi(\|x\|)$$

by (27) and (29), and

$$\begin{aligned} |H_1(\eta_1)| &\geqslant |G_1(\eta_1)| - |F_1(\eta_1)| \\ &> (1 - 2^{-2})\phi(\|\eta_1\|) - 2^{-3}\phi(\|\eta_1\|) > (1 - 2^{-1})\phi(\|\eta_1\|) \end{aligned}$$

by (27) and the fact that equality holds in (29) when $x = \eta_1$.

Now suppose that H_j, ξ_j, η_j have been found for $j = 1, \ldots, m$. By Lemma 4, if r is sufficiently large, then there exists a harmonic polynomial F_{m+1} such that

$$|F_{m+1}| < 2^{-m-4}\phi(0) \quad \text{on } B(0,r), \tag{30}$$

$$\left| F_{m+1}(x) - (P_{m+1}(x - y(r)) - \sum_{j=1}^{m} H_m(x)) \right| < 2^{-m-3}\phi(0) \quad (x \in B(y(r), r)) \tag{31}$$

and

$$|F_{m+1}(x)| < 2^{-m-4}\phi(\|x\|) \quad (x \in \mathbb{R}^n). \tag{32}$$

Let $r > 2\|\eta_m\|$ be so large that such a polynomial F_{m+1} exists and also so large that

$$|(H_1 + \cdots + H_m)(x)| < 2^{-m-4}\phi(\|x\|) \quad (\|x\| \geqslant r). \tag{33}$$

Let $\xi_{m+1} = y(r)$. Then $r = 10^{-1}\|\xi_{m+1}\|$. By Lemma 5, there exists a harmonic polynomial G_{m+1} such that

$$|G_{m+1}| < 2^{-m-4}\phi(0) \quad \text{on } B(0, 2\|\xi_{m+1}\|) \tag{34}$$

and

$$|G_{m+1}(x)| \leqslant (1 - 2^{-m-2})\phi(\|x\|) \quad (x \in \mathbb{R}^n) \tag{35}$$

with equality when $x = \eta_{m+1}$, where η_{m+1} is some point such that $\|\eta_{m+1}\| > 2\|\xi_{m+1}\|$, which we may assume lies on the x_1-axis. Let $H_{m+1} = F_{m+1} + G_{m+1}$. Then, by (30) and (34),

$$|H_{m+1}| \leqslant |F_{m+1}| + |G_{m+1}| < 2^{-m-3}\phi(0) \quad \text{on } B(0, 10^{-1}\|\xi_{m+1}\|),$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{m+1} H_j(x) - P_{m+1}(x - \xi_{m+1})\right| \\
&\quad \leqslant |G_{m+1}(x)| + \left|\sum_{j=1}^{m} H_k(x) + F_{m+1}(x) - P_{m+1}(x - \xi_{m+1})\right| \\
&\quad < 2^{-m-4}\phi(0) + 2^{-m-3}\phi(0) < 2^{-m-2} \quad (x \in B(\xi_{m+1}, 10^{-1}\|\xi_{m+1}\|)),
\end{aligned}$$

by (34) and (31). Also

$$\left|\sum_{j=1}^{m+1} H_j\right| \leqslant \left|\sum_{j=1}^{m} H_j\right| + |F_{m+1}| + |G_{m+1}|.$$

Hence if $\|x\| < 10^{-1}\|\xi_{m+1}\|$, then

$$\left|\sum_{j=1}^{m+1} H_j(x)\right| < (1 - 2^{-m-2})\phi(\|x\|) + 2^{-m-3}\phi(0) < (1 - 2^{-m-3})\phi(\|x\|)$$

by the inductive estimate (24) for $|\sum_{j=1}^{m} H_j|$ and (30) and (34); if $\|x\| \geqslant 10^{-1}\|\xi_{m+1}\|$, then

$$\left|\sum_{j+1}^{m+1} H_j(x)\right| < 2^{-m-4}\phi(\|x\|) + 2^{-m-4}\phi(\|x\|) + (1-2^{-m-2})\phi(\|x\|)$$
$$= (1-2^{-m-3})\phi(\|x\|)$$

by (33), (32) and (35). Finally,

$$\begin{aligned}\left|\sum_{j=1}^{m+1} H_j(\eta_{m+1})\right| &\geqslant |G_{m+1}(\eta_{m+1})| - |F_{m+1}(\eta_{m+1})| - \left|\sum_{j=1}^{m+1} H_j(\eta_{m+1})\right| \\ &\geqslant (1-2^{-m-2})\phi(\|\eta_{m+1}\|) - 2^{-m-3}\phi(\|\eta_{m+1}\|) \\ &> (1-2^{-m-1})\phi(\|\eta_{m+1}\|)\end{aligned}$$

by (32), (33) and the equality that holds in (35) when $x = \eta_{m+1}$. This completes the proof of the Proposition.

3.2.

Here we complete the proof of the Theorem for the case where $n \geqslant 3$. We suppose that the sequence (P_m) of harmonic polynomials in the Proposition is dense in $\mathcal{H}_n$. Let the harmonic polynomials H_m and the points ξ_m, η_m be as in the Proposition. If $R > 0$ and $\|\xi_{m_0}\| > 10R$, then $|H_m| < 2^{-m}\phi(0)$ on $B(0, R)$, whenever $m > m_o$. Hence $\sum_1^\infty H_m$ converges locally uniformly on $\mathbb{R}^n$ to some harmonic function H on $\mathbb{R}^n$. It follows from (24) that $|H(x)| \leqslant \phi(\|x\|)$ for each x in $\mathbb{R}^n$. Also

$$\begin{aligned}|H(\eta_m)| &\geqslant \left|\sum_{j=1}^{m} H_j(\eta_m)\right| - \sum_{j=m+1}^{\infty} |H_j(\eta_m)| \\ &\geqslant (1-2^{-m})\phi(\|\eta_m\|) - \sum_{j=m+1}^{\infty} 2^{-j}\phi(0),\end{aligned}$$

so (2) holds.

It remains to show that H is universal. Let K be a compact set in $\mathbb{R}^n$, let $h \in \mathcal{H}_n$, and let ε be a positive number. We choose an integer m such that $|P_m - h| < \varepsilon/3$ on K, so large that $K \subset B(0, 10^{-1}\|\xi_m\|)$ and $2^{-m} < \varepsilon/3$. If $x \in K$, then $x + \xi_m \in B(\xi_m, 10^{-1}\|\xi_m\|)$, so

$$\begin{aligned}|H(x+\xi_m) - h(x)| &\leqslant |H(x+\xi_m) - P_m(x)| + |P_m(x) - h(x)| \\ &\leqslant \left|\sum_{j=1}^{m} H_j(x+\xi_m) - P_m(x)\right| + \sum_{j=m+1}^{\infty} |H_j(x+\xi_m)| + \varepsilon/3 \\ &< 2^{-m} + \sum_{j=m+1}^{\infty} 2^{-j} + \varepsilon/3 < \varepsilon.\end{aligned}$$

4. The case $n = 2$

4.1.

We indicate here the minor changes that are required to prove the Theorem in the case $n = 2$. We start with a modification of Lemma 2.

Lemma 2′. *There exist positive numbers C_2 and p with the following property. If $r > 2$ and j, k are positive integers, and if u is a harmonic function on $\mathbb{R}^2$ satisfying* (3), *then*

$$d_j(\tilde{u}, T(r, r)) < C_2 r^{k+1} e^{-pj} \log r, \tag{36}$$

where $\tilde{u} = 0$ on $\overline{B(0, r)}$ and $\tilde{u} = u$ on $\overline{B(y(r), r)}$.

Lemma 2′ is proved by making some alterations to the proof of Lemma 2. The first change is in formula (9) defining $v(x)$, as we now explain. In proving (8), we need to replace the fundamental function $\|x - y\|^{2-n}$ on $\mathbb{R}^n \times \mathbb{R}^n (n \geqslant 3)$ by the corresponding function $-\log \|x - y\|$ on $\mathbb{R}^2 \times \mathbb{R}^2$. With this modification, we find, again using [6, Lemma 4.3.6], that (8) holds with $n = 2$, provided we replace the definition (9) by

$$v(x) = -(2\pi)^{-1} \int \log \|x - y\| \, d\mu(y). \tag{37}$$

The estimate (11) for $|v|$ is modified as follows. Since $2 < r \leqslant \|x - y\| \leqslant 13r$ when $x \in \partial T(r, 2r)$ and $y \in T(r, r)$, and since μ is a signed measure of total variation 1 with support contained in $T(r, r)$, we have

$$|v| \leqslant (2\pi)^{-1} \log(13r) \quad \text{on } \partial T(r, 2r). \tag{38}$$

The formula corresponding to (12) is

$$\log \|x - y\| = \log \|x\| + \sum_{m=1}^{\infty} m^{-1} P_m(x, y) \quad (\|x\| > \|y\|), \tag{39}$$

where the functions P_m have the same properties as in the case where $n \geqslant 3$; see e.g. [5, p. 49] or [18, p. 75]. (Explicitly, if $x = (\rho \cos \theta, \rho \sin \theta)$ and $y = (\rho' \cos \phi, \rho' \sin \phi)$, then a calculation shows that $P_m(x, y) = -(\rho'/\rho)^m \cos m(\theta - \phi)$.) Using (37), (39), (7) and the properties of P_m we see that (13) holds in the case $n = 2$. From (38), (13) and Lemma 1, we find that with $n = 2$ an inequality corresponding to (14) is

$$|v(x)| \leqslant (2\pi)^{-1} C_1 \log(13r) e^{-p(j+1)} \quad (x \in \mathbb{R}^2 \backslash T(r, 3r)^{\circ}).$$

Arguing as in the final sentence of Section 2.2 and noting that $\log(13r) < 5 \log r$ when $r > 2$, we find that (36) holds.

We claim that Lemma 4 holds without alteration with $n = 2$. To prove Lemma 4 with $n = 2$ we replace the requirement (19) by the condition

$$C_2 e r^{1-k} \log r < \varepsilon \phi(0)/C_3. \tag{40}$$

Since we must use Lemma 2′ in place of Lemma 2, we find that a factor $\log r$ must be inserted on the right-hand sides of (21) and (22). In view of (40), we see that these modifications of (21) and (22) imply that inequality (23) and the inequality immediately following it hold without alteration in the case $n = 2$. The proof of Lemma 4 for $n = 2$ can now be completed exactly as in the final sentence of Section 2.4. The proof of the Theorem in Section 3 depends only on Lemmas 4 and 5. Lemma 5 is valid for all dimensions $n \geqslant 2$, and we have just seen that Lemma 4 also holds with $n = 2$. The proof of the Theorem for $n = 2$ can now be completed exactly as in Section 3.

4.2.

The existence of slowly growing universal harmonic functions on $\mathbb{R}^2$ can also be deduced from corresponding growth results for universal holomorphic functions. With $\mathbb{R}^2$ and $\mathbb{C}$ identified in the usual way, it is easy to see that the real part of a universal holomorphic function is a universal harmonic function on $\mathbb{R}^2$. Duios Ruis [16] (see also [14]) showed that if ϕ is an entire function given by $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$, where the coefficients a_j are positive and the sequence (a_{j+1}/a_j) is decreasing with limit 0, then there exists a universal entire holomorphic function F such that $|F(z)| = O(\phi(|z|))$ uniformly as $z \to \infty$. The real part of F is a universal harmonic function on $\mathbb{R}^2$ satisfying the same growth condition. Since such functions ϕ can be of arbitrarily slow transcendental growth, this shows that in some respects our Theorem can be improved, at least in the case $n = 2$.

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